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### Information Sharing Games

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*Publication date:*  
2000

[Link to publication in Tilburg University Research Portal](#)

*Citation for published version (APA):*

Slikker, M., Norde, H. W., & Tijs, S. H. (2000). *Information Sharing Games*. (CentER Discussion Paper; Vol. 2000-100). Operations research.

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Center  
for  
Economic Research

No. 2000-100

**INFORMATION SHARING GAMES**

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October 2000

ISSN 0924-7815

# Information sharing games<sup>\*</sup>

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October 9, 2000

## Abstract

In this paper we study information sharing situations. For every information sharing situation we construct an associated cooperative game, which we call an information sharing game. We show that the class of information sharing games coincides with the class of cooperative games with a population monotonic allocation scheme.

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<sup>\*</sup>The authors thank Rodica Brânzei for useful suggestions and comments.

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# 1 Introduction

In this paper we combine two fields of research in game theory, knowledge and cooperative decision making. Knowledge is one of the areas in *non-cooperative* decision making that has received a considerable amount of attention recently. However, less attention has been paid to the relation between knowledge and *cooperative* decision making. In this paper we will associate cooperative games with situations that deal with information sharing. Our main result is that the resulting cooperative games exhaust the class of cooperative games with a population monotonic allocation scheme.

The work of Aumann (1999a,b) has initiated a considerable amount of research dealing with knowledge and information. Here, we follow Aumann (1999a) by assuming that players do not have perfect information on the true state of the world. Additionally, we assume that players have to make decisions, where the outcome of their decisions does depend on the true state of the world. Sharing information between different players who have to make decisions and who have different information might increase joint (expected) profits. Such an information sharing situation naturally results in a cooperative game with transferable utilities. The resulting game will be called an information sharing game.

The approach taken in this paper falls under the general approach of associating cooperative games with specific (economic) situations. Such an approach was also taken by Shapley and Shubik (1969). They studied the class of market games, i.e., games that were derived from an exchange economy. Their main result states that the class of market games coincides with the class of totally balanced games. Many similar and related results have been derived since. Kalai and Zemel (1982b) showed such a relation between flow games and non-negative totally balanced games. Subsequently, in Kalai and Zemel (1982a) they studied several generalized network problems that yield totally balanced games. Non-negative balanced games were shown to be exhausted by flow games with committee control in Curiel et al. (1989). In van den Nouweland et al. (1993) it is shown that monotonic cooperative games are spanning network games. Finally, we mention economies with land. Legut et al. (1994) showed that every cooperative game that results from an economy with land has a population monotonic allocation scheme (cf. Sprumont (1990)). Reijnierse (1995) showed that a subclass of these games, in which initial endowments of the players are disjoint parcels of land with veto control, exhaust the class of games with a population monotonic allocation scheme. The current paper is in the same style as these papers, dealing with information sharing games and cooperative games with a population monotonic allocation scheme.

The set-up of this paper is as follows. Section 2 deals with preliminaries regarding

cooperative games, population monotonic allocation schemes, and information partitions. In section 3 we introduce information sharing situations and associate a cooperative game with each information sharing situation, which we call an information sharing game. The main result, which states that the class of cooperative games with a population monotonic allocation scheme coincides with the class of information sharing games, can be found in section 4. We conclude in section 5 with some remarks on assumptions that are made in this paper and on possible generalizations to a setting without these assumptions.

## 2 Preliminaries

A cooperative game with transferable utility (TU-game) is a pair  $(N, v)$  where  $N = \{1, \dots, n\}$  denotes the set of players and  $v$  is a real-valued function on the family  $2^N$  of all subsets of  $N$  with  $v(\emptyset) = 0$ . The function  $v$  is called the *characteristic function* of the cooperative game  $(N, v)$ .

Sprumont (1990) introduced population monotonic allocation schemes (PMAS): a vector  $(y_{i,S})_{i \in S, S \subseteq N}$  is a *population monotonic allocation scheme* for the cooperative game  $(N, v)$  if it satisfies the following conditions:

- (a)  $\sum_{i \in S} y_{i,S} = v(S)$  for all  $S \subseteq N$ .
- (b)  $y_{i,S} \leq y_{i,T}$  for all  $S, T \subseteq N$  with  $S \subseteq T$  and all  $i \in S$ .

Knowledge of a player can be modeled in several ways. Aumann (1999a) presents and discusses five equivalent formalizations of the idea of knowledge: signal functions, information functions, information partitions, knowledge operators, and knowledge ufields (universal fields). Here we will concentrate on information partitions. The set of all possible states of the world will be denoted by  $\Omega$ . For notational convenience and clarity of exposition we will assume throughout this paper that  $|\Omega| < \infty$ .<sup>1</sup> For example, if the roll of a dice is studied the set of states of the world is the set of possible results of rolling a dice,  $\Omega = \{1, 2, 3, 4, 5, 6\}$ .

An information partition is a partition of  $\Omega$ . A player with some information partition cannot distinguish between the states of the world that belong to the same partition element. For example if a player will be told that the result of rolling a dice is odd or even, his information partition is  $\{\{1, 3, 5\}, \{2, 4, 6\}\}$ . So, he cannot distinguish between states 1, 3, and 5, nor can he distinguish between states 2, 4, and 6. Denote the set of all partitions of  $\Omega$  by  $\Pi$ . For all information partitions  $I \in \Pi$  and all  $\omega \in \Omega$  we denote the element of  $I$  containing  $\omega$  by  $P(\omega, I)$ . For every two information partitions, say  $I$

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<sup>1</sup>See section 5 for some remarks on assumptions made in this paper.

and  $I^*$ , we call  $I$  a *refinement* of  $I^*$  if every element of  $I$  is a subset of an element of  $I^*$ . If  $I$  is a refinement of  $I^*$  then we call  $I^*$  a *coarsening* of  $I$ .

Consider two players, say 1 and 2, with information partitions  $I_1, I_2 \in \Pi$ , respectively. Then, if  $\omega^*$  is the true state of the world, player 1 knows that the true state of the world is an element of  $P(\omega^*, I_1)$ , whereas player 2 knows that the true state of the world is an element of  $P(\omega^*, I_2)$ . So, if they share their information they know that the true state of the world belongs to  $P(\omega^*, I_1) \cap P(\omega^*, I_2)$ .

The joint information partition of these two players, if they share their information, is the coarsest partition of  $\Omega$  that is a refinement of both  $I_1$  and  $I_2$ :

$$I_1 \vee I_2 = \left\{ P_1 \cap P_2 \mid P_1 \in I_1, P_2 \in I_2, P_1 \cap P_2 \neq \emptyset \right\}.$$

The partition  $I_1 \vee I_2$  is called the *join* of  $I_1$  and  $I_2$ . For a set of players  $S$  with information partitions  $(I_i)_{i \in S}$  the join of their information partitions is defined similarly:

$$\bigvee_{i \in S} I_i = \left\{ \bigcap_{i \in S} P_i \mid P_i \in I_i \text{ for all } i \in S, \bigcap_{i \in S} P_i \neq \emptyset \right\}. \quad (1)$$

We assume that all players have the same prior probability measure on the set of possible states of the world. This common prior is denoted by  $\pi$ . Furthermore, we will assume that every state occurs with positive probability. So,  $\pi(\omega) > 0$  for all  $\omega \in \Omega$ . The probability of an event  $A \subseteq \Omega$  is denoted by  $P^\pi(A) = \sum_{\omega \in A} \pi(\omega)$ . Of course,  $P^\pi(\Omega) = 1$ . Once player  $i$  knows that the true state of the world is an element of  $P_i \in I_i$  he can deduce the posterior probabilities from prior probabilities by Bayesian updating:

$$P^\pi(\omega | P_i) = \frac{P^\pi(\{\omega\} \cap P_i)}{P^\pi(P_i)} = \begin{cases} \frac{P^\pi(\{\omega\})}{P^\pi(P_i)} & \text{if } \omega \in P_i; \\ 0 & \text{otherwise.} \end{cases}$$

### 3 The model

In this section we will introduce information sharing situations, which describe economic agents who have to make decisions based on the information they have. The rewards the players receive depend on their own choices and on the true state. We start with an example that illustrates some notions we will encounter in this section. Subsequently, we will formally describe information sharing situations. Finally, we will construct a cooperative game for each information sharing situation.

**Example 3.1** Consider a small hotel with 8 rooms, 4 on the bottom floor and 4 on the top floor. On each floor, two rooms are on the north side and two rooms on the south side. Making additionally use of the east and west side uniquely describes each room in

terms of directions and floors. Three people are in the hotel. All three will sleep in the hotel, but they also know that there is a monster in one of the rooms of the hotel. All three people will get to know partial information about the hiding place of this monster. Player 1 will get to know whether it is on the north or south side, player 2 whether it is on the west or east side, and player 3 will know whether it is on the bottom or top floor. The three players have different preferences. Player 1 wants to be as far away from the monster as possible. Player 2 is interested only in being on a different floor than the monster, while player 3 would like to catch the monster and, hence, wants to be in the same room as the monster. It seems obvious that all players profit from sharing information.  $\diamond$

An *information sharing situation* is a tuple  $(\Omega, N, \pi, (I_i)_{i \in N}, (A_i)_{i \in N}, (r_i)_{i \in N})$ . Here,  $\Omega$  is a (finite) set describing all possible states,  $N = \{1, \dots, n\}$  is the set of players, and  $\pi$  is a probability distribution over  $\Omega$  describing the common prior of the players with a positive probability for all states of the world. For a player  $i \in N$ ,  $I_i$  describes a partition of  $\Omega$ . Two states are in the same partition element in  $I_i$  if player  $i$  cannot distinguish between these states. The set  $A_i$  contains all actions of player  $i \in N$ . We will assume that  $A_i$  is finite for all  $i \in N$ . Finally,  $r_i$  is a function that assigns to every pair  $(\omega, a_i) \in \Omega \times A_i$  a reward  $r_i(\omega, a_i) \in \mathbb{R}$ .

**Example 3.2** Consider the situation described in example 3.1. We will define an information sharing situation describing this example. The set of states in the example consists of all rooms in the hotel, representing the possible places of the monster. Each room can be denoted by a binary code of length 3, where the first coordinate denotes whether the room is on the north side (0) or south side (1) of the hotel. Similarly, the second coordinate denotes whether the room is on the east side (0) or west side (1). The third coordinate denotes whether the room is on the bottom floor (0) or top floor (1). So,  $\Omega = \{0, 1\}^3$  and, for example, 001 refers to the room that is in the north-east corner of the top floor. The three players are contained in player set  $N = \{1, 2, 3\}$ . Beforehand, every player believes that all rooms have the same probability that it contains the monster. Hence, the prior distribution assigns equal probabilities to all rooms, i.e.,  $\pi(\omega) = \frac{1}{8}$  for all  $\omega \in \Omega$ . For notational convenience we will denote the two elements of the information partition of player 1 by  $\mathcal{N}$  and  $\mathcal{S}$ , where  $\mathcal{N}$  contains all rooms on the north side and  $\mathcal{S}$  all rooms on the south side. Similarly, we introduce  $\mathcal{W}$  and  $\mathcal{E}$  for the west and east side. Finally, we denote  $\mathcal{B}$  and  $\mathcal{T}$  for the rooms on the bottom and top

floor, respectively. Now, the information partitions of the players are easily described,

$$I_1 = \{\mathcal{N}, \mathcal{S}\}; \quad (2)$$

$$I_2 = \{\mathcal{E}, \mathcal{W}\}; \quad (3)$$

$$I_3 = \{\mathcal{B}, \mathcal{T}\}. \quad (4)$$

It remains to describe the possible actions and the reward functions of the players. Each player chooses a room to sleep, implying that the action set of each player coincides with the set of possible states, i.e.,  $A_i = \Omega$  for all  $i \in N$ . Finally, the following reward functions capture the preferences of the players,

$$r_1(\omega, a) = 36 \sum_{i=1}^3 |a_i - \omega_i|; \quad (5)$$

$$r_2(\omega, a) = 18 |a_3 - \omega_3|; \quad (6)$$

$$r_3(\omega, a) = \begin{cases} 60 & \text{if } a = \omega; \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

◇

A player will get to know only the element of his information partition containing the true state. This implies that a player will get to know the true state only if this state is the single element of an information partition element. Together with the prior probability distribution over the set of states the player can determine the posterior probability distribution over the elements of the partition element, which is a conditional distribution. Assuming risk-neutrality of the player, this player will choose the strategy that maximizes his expected payoff. A priori, a player can already make up his mind what his choice will be, for any possible element of his information set. The a-priori-strategy of player  $i$  can formally be described by a function  $x_i$  from the set of states  $\Omega$  into the set of strategies  $A_i$ . Since a player cannot distinguish between states in the same partition element, this function has to be constant on every element of his information partition, i.e., on every  $P_i \in I_i$ . The set of all such functions for player  $i$  will be denoted by  $X_i(A_i, I_i)$ . Since  $\Omega$  is a finite set it follows that  $I_i$  contains a finite number of partition elements. Together with the finiteness of  $A_i$  it follows that the set  $X_i(A_i, I_i)$  is finite. The (a-priori-)value of player  $i$ , notation  $v^{I_i}(\{i\})$ , is defined as his maximum expected payoff, i.e.,

$$v^{I_i}(\{i\}) = \max_{x_i \in X_i(A_i, I_i)} E_\pi[r_i(\cdot, x_i(\cdot))], \quad (8)$$

where  $E_\pi$  represents the expected value with respect to prior distribution  $\pi$ . This maximum is well defined by the finiteness of  $X_i(A_i, I_i)$ .



**Example 3.3** Consider the information sharing situation of example 3.2. Player 1 knows whether the monster is in the north or the south side of the hotel. Given the reward function of player 1, an optimal action of player 1 is to choose an arbitrary room in the south side if the monster is in the north side and an arbitrary room in the north side if the monster is in the south side. Then the average 'distance' between player 1 and the monster is  $\frac{1}{4} \star 1 + \frac{2}{4} \star 2 + \frac{1}{4} \star 3 = 2$ . Hence, the expected payoff to player 1 is  $v^{I_1}(\{1\}) = 2 \star 36 = 72$ .  $\diamond$

For any information sharing situation we can determine the value of a player. However, players might have an incentive to cooperate with each other by sharing information. Sharing information can result in higher joint profits for the players. We will construct a cooperative game with each information sharing situation, representing the idea that cooperation between the players consists of sharing information. Sharing information between the players results in a refined information partition for the players that share the information. If players  $i$  and  $j$  share their information, both end up with information partition  $I_i \vee I_j$ . With this information player  $i$  can expect value  $v^{I_i \vee I_j}(\{i\})$ . So, together players  $i$  and  $j$  can expect

$$v^{I_i \vee I_j}(\{i\}) + v^{I_i \vee I_j}(\{j\}).$$

Obviously, this idea can be extended to every set of players. This results in a cooperative game which we call the *information sharing game*  $(N, v^\kappa)$  associated with information situation  $\kappa$ , where

$$v^\kappa(S) = \sum_{i \in S} v^{\vee_{j \in S} I_j}(\{i\}) \text{ for all } S \subseteq N.$$

**Example 3.4** Denote the information sharing situation that was studied in examples 3.2 and 3.3 by  $\kappa$ . We already showed that  $v^\kappa(\{1\}) = 72$ . Now, consider coalition  $\{1, 3\}$ . After players 1 and 3 have shared their information, player 1 knows not only whether the monster is in the north or south side of the hotel, but also whether it is on the bottom floor or the top floor. Hence, player 1 will choose the other side and other floor. Additionally, he will make an arbitrary choice between the east side and the west side. Hence, the average distance between player 1 and the monster will be  $\frac{1}{2} \star 2 + \frac{1}{2} \star 3 = 2\frac{1}{2}$ , implying that the expected contribution of player 1 to  $v(\{1, 3\})$  is  $2\frac{1}{2} \star 36 = 90$ . Since player 3 has a 50% chance of choosing the correct room it follows that the expected contribution of player 3 to  $v^\kappa(\{1, 3\})$  is  $\frac{1}{2} \star 0 + \frac{1}{2} \star 60 = 30$ . We conclude that  $v^\kappa(\{1, 3\}) = 90 + 30 = 120$ .

Similar calculations for the other coalitions show that the information sharing game

associated with  $\kappa$  is described by

$$v^\kappa(T) = \begin{cases} 0 & \text{if } T = \emptyset; \\ 72 & \text{if } T = \{1\}; \\ 9 & \text{if } T = \{2\}; \\ 15 & \text{if } T = \{3\}; \\ 99 & \text{if } T = \{1, 2\}; \\ 120 & \text{if } T = \{1, 3\}; \\ 48 & \text{if } T = \{2, 3\}; \\ 186 & \text{if } T = N. \end{cases} \quad (9)$$

◇

## 4 Population monotonic allocation schemes

In this section we show that the class of information sharing games coincides with the class of cooperative games that have a population monotonic allocation scheme. Firstly, consider the following example.

**Example 4.1** Consider example 3.4. It is easily checked that  $(v^{\vee_{j \in S} I_j}(\{i\}))_{i \in S, S \subseteq N}$  is a population monotonic allocation scheme for the game  $(N, v^\kappa)$ . This population monotonic allocation scheme is represented in table 1.

|           | 1   | 2  | 3  |
|-----------|-----|----|----|
| $\{1\}$   | 72  | *  | *  |
| $\{2\}$   | *   | 9  | *  |
| $\{3\}$   | *   | *  | 15 |
| $\{1,2\}$ | 90  | 9  | *  |
| $\{1,3\}$ | 90  | *  | 30 |
| $\{2,3\}$ | *   | 18 | 30 |
| $N$       | 108 | 18 | 60 |

Table 1: A population monotonic allocation scheme for the game  $(N, v^\kappa)$  in example 4.1.

According to this population monotonic allocation scheme player 2, for example, receives 18 if coalition  $\{2, 3\}$  would happen to result. ◇

Example 4.1 describes a direct way to determine a population monotonic allocation scheme in an information sharing game, using the structure of the underlying information sharing situation. The following theorem shows that this holds in general.

**Theorem 4.1** Let  $\kappa = (-, N, \pi, (I_i)_{i \in N}, (A_i)_{i \in N}, (r_i)_{i \in N})$  be an information sharing situation and  $(N, v^\kappa)$  the associated information sharing game. Then  $(v^{\vee_{j \in S} I_j}(\{i\}))_{i \in S, S \subseteq N}$  is a population monotonic allocation scheme for the game  $(N, v^\kappa)$ .

**Proof:** We will check that  $(v^{\vee_{j \in S} I_j}(\{i\}))_{i \in S, S \subseteq N}$  satisfies the conditions in the definition of population monotonic allocation schemes.

- (a) Let  $S \subseteq N$ . Then  $\sum_{i \in S} v^{\vee_{j \in S} I_j}(\{i\}) = v^\kappa(S)$  by definition of  $v^\kappa(S)$ .
- (b) Let  $S, T \subseteq N$  with  $S \subseteq T$ . Then  $\vee_{j \in S} I_j$  is coarser than  $\vee_{j \in T} I_j$ , i.e., every element of  $\vee_{j \in S} I_j$  is partitioned in one or more elements of  $\vee_{j \in T} I_j$ , since  $\vee_{j \in T} I_j = (\vee_{j \in S} I_j) \vee (\vee_{j \in T \setminus S} I_j)$ . This implies that every function that is constant on every element of  $\vee_{j \in S} I_j$  is also constant on every element  $\vee_{j \in T} I_j$ :

$$X_i(A_i, \vee_{j \in S} I_j) \subseteq X_i(A_i, \vee_{j \in T} I_j) \text{ for all } i \in S.$$

Hence, for all  $i \in S$  it holds that

$$\begin{aligned} v^{\vee_{j \in S} I_j}(\{i\}) &= \max_{x_i \in X_i(A_i, \vee_{j \in S} I_j)} E_\pi[r_i(\cdot, x_i(\cdot))] \\ &\leq \max_{x_i \in X_i(A_i, \vee_{j \in T} I_j)} E_\pi[r_i(\cdot, x_i(\cdot))] \\ &= v^{\vee_{j \in T} I_j}(\{i\}). \end{aligned}$$

This completes the proof. □

Theorem 4.1 implies that every information sharing game has a PMAS. In the following theorem we will deal with the the reverse question, i.e., is every game with a PMAS an information sharing game? We show that this question can be answered in the affirmative.

**Theorem 4.2** Let  $(N, v)$  be a cooperative game with a population monotonic allocation scheme. Then there exists an information sharing situation  $\kappa$  such that  $v^\kappa = v$ .

**Proof:** Let  $(y_{i,S})_{i \in S, S \subseteq N}$  be a population monotonic allocation scheme for  $(N, v)$ . Define the information sharing situation  $\kappa = (-, N, \pi, (I_i)_{i \in N}, (A_i)_{i \in N}, (r_i)_{i \in N})$  in the following way.

- 1) The set  $\Omega$  is the collection of all maps  $\omega : 2^N \rightarrow 2^N$  that satisfy  $\omega(S) \subseteq S$  for every  $S \in 2^N$ . A state of the world  $\omega$  can be seen as a mechanism that assigns to every coalition a selected subgroup.
- 2) The prior probability distribution  $\pi$  is defined by  $\pi(\omega) = 1/|\Omega|$  for every  $\omega \in \Omega$ . So  $\pi$  assigns equal probabilities to all states in  $\Omega$ .
- 3) In order to define the information partition  $I_i$  of player  $i \in N$  we introduce the equivalence relation  $\sim_i$  on  $\Omega$ :  $\omega \sim_i \omega'$  if for all  $S \subseteq N$  with  $i \in S$  we have  $i \in \omega(S)$  iff  $i \in \omega'(S)$ . The information partition  $I_i$  consists of all equivalence classes of  $\sim_i$ . The information partition  $I_i$  can be interpreted as follows: player  $i$  only knows for every coalition to which he belongs whether he is part of the selected subgroup or not.
- 4) The action set  $A_i$  of player  $i \in N$  is defined by  $A_i = \{S \subseteq N \mid i \in S\} \times \{0, 1\}$ . An action  $a = (a_1, a_2)$  of player  $i$  consists of a choice of a coalition  $a_1$  to which he belongs together with a guess  $a_2$  whether the size of the selected subgroup  $\omega(a_1)$  is even ( $a_2 = 0$ ) or odd ( $a_2 = 1$ ).
- 5) The reward function  $r_i$  of player  $i \in N$  is defined by

$$r_i(\omega, a) = \begin{cases} y_{i,a_1} & \text{if } |\omega(a_1)| \bmod 2 = a_2; \\ 2y_{i,\{i\}} - y_{i,a_1} & \text{otherwise} \end{cases}$$

for every  $\omega \in \Omega$  and  $a \in A_i$ . So, player  $i$  receives  $y_{i,a_1}$  if he guesses correctly whether the size of the selected subgroup of the coalition which he chose ( $a_1$ ) is even or odd. Otherwise, he receives the smaller amount  $2y_{i,\{i\}} - y_{i,a_1}$ .

In order to show that  $v^\kappa = v$  it suffices to prove for every  $S \subseteq N$ ,  $i \in S$  that  $v^{\vee_{j \in S} I_j}(\{i\}) = y_{i,S}$ . Since then we have, for every  $S \subseteq N$ ,

$$v^\kappa(S) = \sum_{i \in S} v^{\vee_{j \in S} I_j}(\{i\}) = \sum_{i \in S} y_{i,S} = v(S).$$

So, let  $S \subseteq N$ ,  $i \in S$ . Let  $P \in \vee_{j \in S} I_j$ . Note that, for every  $T \subseteq S$ , the selected subgroup does not depend upon  $\omega \in P$ , i.e. for every  $\omega, \omega' \in P$  we have  $\omega(T) = \omega'(T)$ . We will show that the maximal conditional expected reward for player  $i$ , conditional upon  $P$ , equals  $y_{i,S}$ . Thereby we distinguish three cases.

First, consider the action  $a^* = (a_1^*, a_2^*) \in A_i$  with  $a_1^* = S$  and  $a_2^* = |\omega(S)| \bmod 2$  for some  $\omega \in P$ . Note that  $a_2^*$  is well-defined since  $\omega(S) = \omega'(S)$  for every  $\omega, \omega' \in P$ . Clearly,  $r_i(\omega, a^*) = y_{i,S}$  for every  $\omega \in P$ , so the conditional expected reward for player  $i$ , if he chooses action  $a^*$ , equals  $y_{i,S}$ .

Secondly, consider action  $a = (a_1, a_2)$  with  $a_1 \subsetneq S$ . Keep in mind that  $\omega(a_1) = \omega'(a_1)$

for every  $\omega, \omega' \in P$ . If  $a_2 = |\omega(a_1)| \bmod 2$  for every  $\omega \in P$  then  $r_i(\omega, a) = y_{i,a_1}$  for every  $\omega \in P$  and hence the conditional expected reward for player  $i$ , if he chooses action  $a$ , equals  $y_{i,a_1} \leq y_{i,S}$ . If  $a_2 \neq |\omega(a_1)| \bmod 2$  for every  $\omega \in P$  then  $r_i(\omega, a) = 2y_{i,\{i\}} - y_{i,a_1}$  for every  $\omega \in P$  and hence the conditional expected reward for player  $i$ , if he chooses action  $a$ , equals  $2y_{i,\{i\}} - y_{i,a_1} \leq y_{i,a_1} \leq y_{i,S}$ .

Thirdly, consider an action  $a = (a_1, a_2) \in A_i$  with  $a_1 \not\subseteq S$ . Let  $j \in a_1 \setminus S$ . With every  $\omega \in \Omega$  we can associate  $\omega' \in \Omega$ ,  $\omega \neq \omega'$  in the following way:  $\omega'(T) = \omega(T)$  for every  $T \neq a_1$ ,  $\omega'(a_1) = \omega(a_1) \setminus \{j\}$  if  $j \in \omega(a_1)$  and  $\omega'(a_1) = \omega(a_1) \cup \{j\}$  if  $j \notin \omega(a_1)$ . Note that  $\omega' \sim_i \omega$  for every  $i \in N \setminus \{j\}$ . In particular  $\omega' \sim_i \omega$  for every  $i \in S$ . Therefore, we have  $\omega \in P$  iff  $\omega' \in P$ . Moreover,  $|\omega(a_1)|$  is even iff  $|\omega'(a_1)|$  is odd. Since  $(\omega')' = \omega$  for every  $\omega \in \Omega$  it follows that for half of the states  $\omega \in P$  we get  $r_i(\omega, a) = y_{i,a_1}$  and for the other half of the states  $\omega \in P$  we get  $r_i(\omega, a) = 2y_{i,\{i\}} - y_{i,a_1}$ . So, the conditional expected reward for player  $i$ , if he chooses action  $a$ , equals  $\frac{1}{2}y_{i,a_1} + \frac{1}{2}(2y_{i,\{i\}} - y_{i,a_1}) = y_{i,\{i\}} \leq y_{i,S}$ . We conclude that the maximal conditional expected reward for player  $i$  equals  $y_{i,S}$ . Since this is true for every  $P \in \bigvee_{j \in S} I_j$  we get

$$v^{\bigvee_{j \in S} I_j}(\{i\}) = y_{i,S}.$$

This finishes the proof.  $\square$

## 5 Remarks

We conclude in this section with some remarks on assumptions made in the paper. For several reasons we have assumed finiteness of several sets in our model. Firstly, because of notational convenience and clarity of exposition. Extending to a setting with  $\Omega$  possibly being infinite demands for the use of a  $\sigma$ -algebra of subsets of  $\Omega$ , a probability measure on this  $\sigma$ -algebra, and a reward function that is bounded and measurable with respect to this  $\sigma$ -algebra. In our opinion this will distract attention from the main ideas in this paper, though extending our model in this direction is very well possible.

Secondly, non-finiteness of the number of actions of a player would demand for a slight change in the model, by taking the supremum rather than the maximum in (8) or by making some additional assumptions on  $A_i$  and  $r_i$ , e.g., compactness of  $A_i$  and continuity of  $r_i$ . Again we note that such a generalization is possible.

We remark that for details on extensions as described above, the interested reader can turn to Brânzei et al. (2000), who introduce a model that is similar to our model, but with one decision maker only. Brânzei et al. (2000) do not make any finiteness-assumptions.

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